

# Multivacuum States in a Fermionic Gap Equation with massive gluons and confinement

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We study the nontrivial solutions of the QCD fermionic gap equation including the contribution of dynamically massive gluons and the confining propagator proposed by Cornwall. Without the confining propagator, in the case of non-running gluon mass ( $m_g$ ), we found the multivacuum solutions (replicas) reported in the literature and we were able to define limits on  $m_g$  for dynamical chiral symmetry breaking. On the other side, when considering the running in the gluon mass the vacuum replicas are absent in the limits on  $m_g$  where the chiral symmetry is broken. In the pure confining sector, the multivacuum states are always absent so it is said that only one stable solution for the gap equation is found as claimed in previous analysis using different approaches. Finally in the case of the complete gap equation i.e. with both contributions, the vacuum replicas are also absent in both cases; with constant and with running gluon mass.

## I. INTRODUCTION

In Quantum Chromodynamics (QCD) the fundamental degrees of freedom of the theory are not detected as free objects and the quark self-energy can drive the appearance of a dynamical mass. These two phenomena are known as confinement of quarks and gluons and dynamical chiral symmetry breaking (CSB), respectively. When studied separately, both phenomena are partially understood: For the latter, the idea is well accepted that the chiral condensate obtain a nontrivial vacuum expected value leading to the generation of a non-zero dynamical quark mass. In this scheme, the (pseudo)Goldstone bosons associated with the breaking of the continuous symmetry are the pions. One theoretical tool used to study this process is the fermionic gap equation (FGE) which can be obtained from the Schwinger-Dyson equations (SDE) for the fermionic fields [1]. In the case of confinement, an order parameter used to describe the transition from the confined to the deconfined phase is the vacuum expectation value of the Polyakov loop  $L$  [2, 3]. Although this description is well suited only for pure gluons QCD, there are some modern approaches with the aim of including quarks [4, 5].

Despite the advances in the understanding of both phenomena, one of the actual challenges for a complete description of the nonperturbative QCD regime is the connection between those important phases of the IR behavior of QCD. For example, it has been found that the deconfinement transition and the chiral symmetry restoration occur approximately at the same temperature for quarks in the fundamental representation [6, 7], which is different for the adjoint representation [8, 9]. The analysis of this behavior has been recently explored [10] in the framework of the gap equation with the inclusion of Cornwall's confining propagator which has been shown to provide a good description of the discrepancy between the chiral transition of fundamental and adjoint quarks. Another issue which concerns the relation of confinement and chiral symmetry breaking is the idea that removal of

central vortices, may or may not impact in the recovery of the chiral symmetry. It was found that at least for  $SU(2)$  this condition is satisfied [11], however calculations for  $SU(3)$  are not yet conclusive [12].

The authors in reference [13], using a Hamiltonian approach to QCD in Coulomb gauge, report that the two dimensional QCD possesses only one possible vacuum state, given by the solution of the mass-gap equation, while the four-dimensional theory possesses an excited vacuum replica. Those results and a theoretical framework are explored in successive works [14, 15]. These authors also suggest that for the pure linearly rising potential, the interaction is not strong enough to hold any replicas so that “only one chirally nonsymmetric solution to the mass-gap equation may exist” [13]. Furthermore, the authors of reference [16] studied the fermionic gap equation for pure QCD and they argue that the excited vacuum states are a consequence of the nature of the gap equation, since it is an integral equation. However, they also show that this vacuum states do not affect what we know about the hadronic spectrum. A quite similar analysis is performed in reference [17] but for  $QED_3$  in which oscillatory solutions are found for the gap equation, solutions which are characterized by the number of zeros.

Nowadays the idea that nonperturbative effects can drive massive propagators for the gauge bosons as suggested by Cornwall [18] is well accepted, especially because this result has been confirmed by lattice simulations [19, 20] and modern approaches using the Dyson-Schwinger equations [21, 22]. The consequence of the inclusion of massive gluons in the analysis of chiral symmetry breaking has been well explored, so that it is known that for the accepted value [18] of the dynamical gluon mass  $m_g \approx 2\Lambda_{QCD}$  (being  $\Lambda_{QCD}$  the QCD scale), the fermionic gap equation is too weak to allow the CSB for quarks in the fundamental representation [23–25]. Also, the positivity issues discussed in reference [26] show that  $m_g > 1.2\Lambda_{QCD}$ , values for which CSB is not yet achieved with the standard fermionic gap equation. As a solution to this issue, Cornwall proposed [27] a modification of

the Mandelstam confining propagator [28] which behaves like  $1/k^4$  for  $1/(k^2 + m^2)^2$ . Here,  $m$  is a parameter necessary for entropic reasons, i.e. space-time fluctuations in the Wilson loop, which turns out to be approximately equal to the dynamical quark mass at zero momentum ( $m \approx M$ ) in order to allow the formation of massless bound states [27, 29]. With this form, the confining propagator leads us to a satisfactory phenomenological interquark potential, and it is also free of IR singularities. This propagator has to be introduced by hand, being a result of vortices which appear when the gluon acquires a dynamical mass [29, 30].

In this paper we explore more about the inclusion of the confining propagator proposed by Cornwall in the fermionic gap equation. We calculate numerical solutions (for different values of the entropic parameter  $m$ ) to study the connection between confinement and CSB. We also check whether or not there are multivacuum states for this equation. In section II we present the complete gap equation and the parameters in which it depends. In section III we solve the gap equation for the one-gluon sector and we found limits on  $m_g$  below which the CSB occurs. Section IV presents a similar analysis for the confining gap equation and section V does the same for the complete gap equation. We finally present our summary and conclusions.

## II. FERMIONIC GAP EQUATION

The dynamical quark mass  $M(p^2)$  can be obtained from the so called complete gap equation, which is the integral equation

$$M(p^2) = \int \frac{d^4k}{(2\pi)^4} \{G_c + G_g\} \frac{M(k^2)}{k^2 + M^2(k^2)}, \quad (1)$$

with Kernels  $G_c$  and  $G_g$ . These correspond to the confining and one-gluon contribution, given by

$$G_c(p-k) = \frac{32\pi K_F}{[(p-k)^2 + m^2]^2}, \quad (2)$$

$$G_g(p-k) = \frac{3C_2 \bar{g}^2(p-k)}{(p-k)^2 + m_g^2(k^2)}, \quad (3)$$

where  $K_F$  is the string tension,  $m$  the mentioned entropic parameter,  $C_2$  the Cassimir eigenvalue,  $m_g(k)$  the dynamical gluon mass and  $\bar{g}(k)$  the effective charge given by [18]

$$\bar{g}^2(k^2) = \frac{1}{b \ln \left[ \frac{k^2 + 4m_g^2(k^2)}{\Lambda_{QCD}^2} \right]}, \quad (4)$$

where  $b = (33 - 2n_f)/48\pi^2$  is the one-loop coefficient in the beta function with  $n_f$  flavors.

Equation (1) can be simplified using the angular approximations (discussed in [31], [23] and [27]), so that

$$f(x) = \int_0^\infty dy [F(x)\theta(x-y) + (x \leftrightarrow y)] \frac{yf(y)}{y + f^2(y)}, \quad (5)$$

with

$$F(x; \rho, \gamma) = \left\{ \frac{ag(x)}{[x + \gamma \tilde{m}_g^2(x)]} + \frac{\rho}{(x + \rho/\rho_c)^2} \right\}, \quad (6)$$

and

$$g(x) = \ln^{-1} \beta [x + 4\gamma \tilde{m}_g^2(x)]. \quad (7)$$

Here we have used the definitions:  $x = p^2/M^2$ ,  $y = k^2/M^2$ ,  $f(x) = M(x)/M$ ,  $g(x) = b\bar{g}^2(x)$ ,  $a = 3C_2/16\pi^2 b$ ,  $\beta = M^2/\Lambda_{QCD}^2$ ,  $\rho = 2K_F/\pi M^2$ ,  $\rho_c = 2K_F/\pi m^2$ ,  $m_g = m_g(0)$ ,  $\tilde{m}_g(x) = m_g(x)/m_g$  and  $\gamma = m_g^2/M^2$ .

Equation (5) can be transformed into the boundary value problem

$$F'(x)f''(x) - F''(x)f'(x) - [F'(x)]^2 \frac{xf(x)}{x+f^2(x)} = 0 \quad (8)$$

$$f(0) = 1 \quad \text{and} \quad f'(0) = 0,$$

with an extra IR condition (given by equation (5))<sup>1</sup>

$$1 = \int_0^\infty dy F(y; \rho, \gamma) \frac{yf(y)}{y + f^2(y)} = I(\rho, \gamma). \quad (9)$$

The values of  $(\rho, \gamma)$  which satisfy the previous condition, correspond to the bifurcation points of the integral equation (9) and with them we can find the dynamical quark mass  $M$ .

In the successive sections, we solve the boundary-value problem (8) using the IR condition (9) and also considering the entropic condition  $m \approx M$  in four cases: only with the one-gluon sector; first with constant and then with running gluon mass; only with the confining sector; and finally with both contributions.

## III. FERMIONIC GAP EQUATION WITH DYNAMICAL GLUON MASS

With no confining contributions, the problem (8) basically stays the same. The only difference is that now

$$F(x; \rho, \gamma) \equiv F_g(x; \gamma) = \frac{ag(x)}{[x + \gamma \tilde{m}_g^2(x)]}, \quad (10)$$

and the IR condition reads <sup>2</sup>

$$1 = \int_0^\infty dy F_g(y; \gamma) \frac{yf_g(y)}{y + f_g^2(y)} = I_g(\gamma). \quad (11)$$

<sup>1</sup> Here we are using  $m_g = 2\Lambda_{QCD}$  so that  $\beta\gamma = 4$ .

<sup>2</sup> Where the subscripts  $g$  are only to identify that the gap equation has only the gluon contribution.

### A. Constant gluon mass

In the case of a nonrunning dynamical gluon mass ( $\tilde{m}_g(x) = 1$ ), the IR condition (11) is not satisfied for the accepted phenomenological value  $m_g \approx 600\text{MeV}$  (FIG. 1). This situation is modified when considering lower values for  $m_g$ . For example, we can see that for values lower than  $172\text{MeV}$ , the IR condition starts to be satisfied, so a condition for CSB to occur is  $m_g \lesssim 172\text{MeV}$ . Another constraint for the gluon mass is if  $m_g < 150\text{MeV}$ , in which case the solution of the problem (8) starts to be unbounded (because of the divergent values of the effective charge given at low momenta). So, the dynamical chiral symmetry breaking for quarks in the fundamental representation is constrained to the values for the dynamical gluon mass (with the corresponding dynamical quark mass) given by

$$\begin{aligned} 150 &\lesssim m_g(\text{MeV}) \lesssim 172, \\ 190 &\gtrsim M(\text{MeV}) \gtrsim 0, \end{aligned} \quad (12)$$

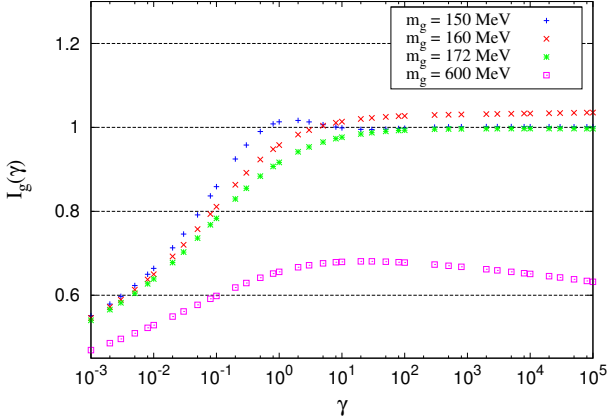


FIG. 1. IR condition (11) for  $n_f = 2$ ,  $\Lambda_{QCD} = 300\text{MeV}$  and different values of  $m_g$ .

For values within this condition, we can see how the multiple vacuum states (the fundamental plus excitations) starts to appear. These correspond to the points in which the condition  $I_g(\gamma) = 1$  is satisfied. For example, for  $m_g = 160\text{MeV}$  we find only one vacuum state, however for a value like  $150\text{MeV}$ , there are three which correspond to  $\gamma = 0.623, 8.797, 207.3$  and  $M = 190, 51, 10\text{MeV}$ . The solutions of the fermionic gap equation 1 (with  $G_c \rightarrow 0$ ) which corresponds to the vacuum state and the two replicas are shown in FIG. 2 for  $m_g = 150\text{MeV}$ . As we can see, the solutions can be classified by the number of zeros as is done in [17].

### B. Running gluon mass

The dynamical gluon mass term has the form [32]

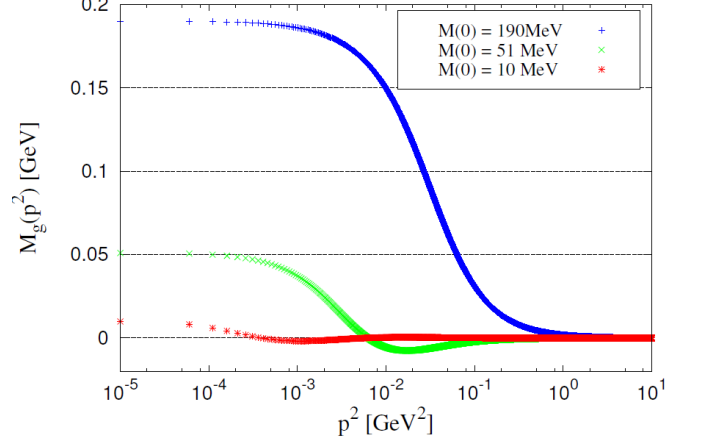


FIG. 2. Dynamical quark mass as a function of the momentum for the fundamental vacuum and two replicas. This function is the solution of the fermionic gap equation (1) with no confining propagator ( $G_c \rightarrow 0$ ) and using  $n_f = 2$  and  $m_g = 150\text{MeV}$ .

$$\tilde{m}_g^2(k^2) = \left[ \ln \left( \frac{k^2 + \mu m_g^2}{\Lambda_{QCD}^2} \right) / \ln \left( \frac{\mu m_g^2}{\Lambda_{QCD}^2} \right) \right]^{-1-\delta}, \quad (13)$$

where  $m_g^2$ ,  $\mu$  and  $\delta$  are parameters whose values are chosen to fit the lattice data. We are going to use the phenomenological values  $m_g = 2\Lambda_{QCD}$ ,  $\mu = 4$  and  $\delta = 1/11$ .

This time, the condition for CSB is:

$$\begin{aligned} 177 &\lesssim m_g(\text{MeV}) \lesssim 204, \\ 310 &\gtrsim M(\text{MeV}) \gtrsim 0. \end{aligned} \quad (14)$$

Within this interval, the gap equation has a single solution as we can see in FIG. 3 where the function  $I_g(\gamma)$  intercepts the line  $I_g = 1$  (i.e. the IR condition (11) is satisfied) at exactly one point. This result is our basis for the statement that when considering the running in the gluon mass the replicas are absent in the limits of  $m_g$  where the chiral symmetry is broken. In this case, the solution of the gap equation has exactly the same shape as in FIG. 2 for  $M(0) = 190\text{MeV}$ , but now  $M(0) = 310\text{MeV}$ .

## IV. FERMIONIC GAP EQUATION WITH CONFINING PROPAGATOR

In this case

$$F(x; \rho, \gamma) \equiv F_c(x; \rho) = \frac{\rho}{(x + \rho/\rho_c)^2}, \quad (15)$$

so this time the free parameter in the IR condition is  $\rho$  and

$$1 = \int_0^\infty dy F_c(y; \rho) \frac{y f_c(y)}{y + f_c^2(y)} = I_c(\rho). \quad (16)$$

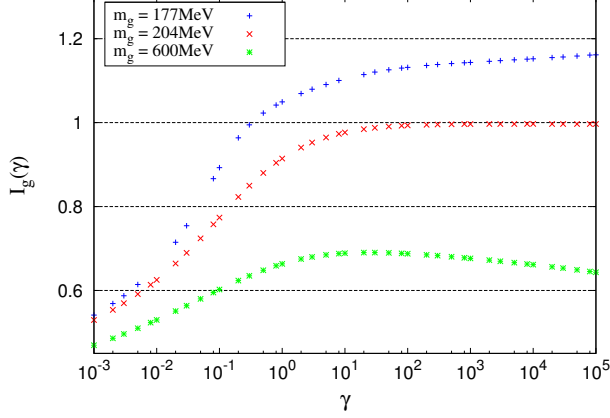


FIG. 3. IR condition with running gluon mass ( $n_f = 2$  and  $\Lambda_{QCD} = 300\text{MeV}$ ).

For values of  $m$  higher than  $272\text{MeV}$  the IR condition is not satisfied, so that we can say that there is no CSB in that regime. For values below this limit, we see (FIG. 4) how the curve intercepts the line  $I_c(\rho) = 1$  at exactly one point, meaning that there is only one vacuum state as it was called before.

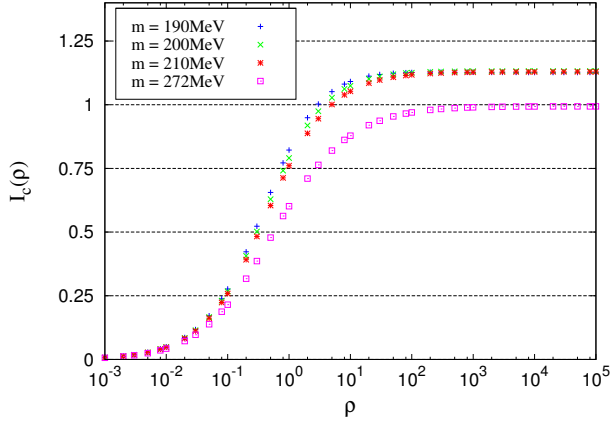


FIG. 4. IR condition for the confining equation with  $K_F = 0.21\text{GeV}^2$  and different values of  $m$ .

Around the value  $m = 200\text{MeV}$ , the resulting dynamical mass  $M$  seems to be close to the value of  $m$  (TABLE I). The value for which both the IR condition (16) and the entropic condition are (best) satisfied is  $m = 196\text{MeV}$  for which  $M = 199\text{MeV}$ .

$m$ (MeV)	$\rho$	$M$ (MeV)	$\alpha = m/M$
185	2.614	226	0.82
190	2.933	213	0.89
<b>196</b>	<b>3.391</b>	<b>199</b>	<b>0.98</b>
200	3.754	189	1.06
205	4.290	178	1.15

TABLE I. Relation between the parameter  $m$ ,  $M$ ,  $\rho$  and  $\alpha$ .

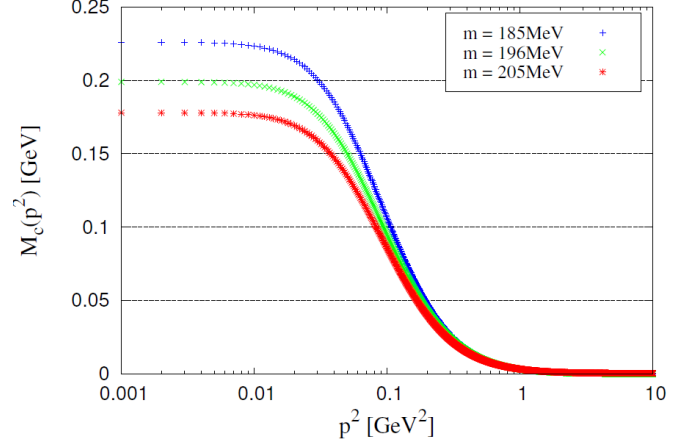


FIG. 5. Dynamical quark mass as a function of the momentum for different values of the entropic parameter  $m$ . This function is the solution of the fermionic gap equation (1) with no one-gluon-exchange propagator ( $G_g \rightarrow 0$ ) and using  $K_F = 0.21\text{GeV}$ .

To finish this section, we show the dynamical masses corresponding to different values of  $m = 190, 200$  and  $210\text{MeV}$ . We can see (FIG. 5) how the solutions are bounded and well behaved (non-oscillatory and non-negative in the domain).

## V. FERMIONIC GAP EQUATION - THE COMPLETE CASE

In our last case we are going to use the complete equation (5), making use of (6), but with the relation  $\gamma = \rho/\rho_g$  where  $\rho_g = 2K_F/\pi m_g^2$ , so that  $\rho$  is the only free parameter in the IR condition. For the complete equation, the situation turns out to be very similar to the one with the confining equation. The values for the dynamical quark mass and the entropic parameter are collected in TABLE II. The IR condition is satisfied for only one value of the free parameter  $\rho$ , so we can say that there are no vacuum replicas. The shape of the function  $I(\rho)$  is similar to the one shown in FIG. 3 (or 4). This time there are no constraints on the dynamical gluon mass for CSB because the confining contribution is driving most of the amount of the dynamical quark mass (as was noticed in [27]). The solutions for the gap equation  $M(p^2)$  are similar to those presented in FIG. 5, i.e. also bounded and well behaved.

$m$ (MeV)	$\rho$	$M$ (MeV)	$\alpha = m/M$
200	2.443	234	0.85
209	2.934	213	0.98
<b>210</b>	<b>2.997</b>	<b>211</b>	<b>1.00</b>
211	3.062	209	1.01
220	3.743	189	1.16

TABLE II. Dynamical mass for the complete gap equation.

## SUMMARY AND CONCLUSIONS

We studied the fermionic gap equation with the inclusion of dynamical massive gluons. We found that there is no CSB for the accepted value of the dynamical gluon mass  $m_g$ . We also found limits on  $m_g$  for dynamical chiral symmetry breaking in both cases; with constant and running gluon mass. The former case shows the appearance of the so called vacuum replicas, which correspond to higher order bifurcation points in the gap equation. For the running case, the replicas are absent in the limits where CSB is founded. We also studied the QCD-like gap equation with the confining propagator proposed by Cornwall. In this case we found CSB for values of the entropic parameter  $m$  compatibles with the entropic condition  $m \approx M$ . We finally studied the complete gap equation which combines both contributions. In this last case the replicas are also absent and CSB is present even for higher values of the gluon mass compatibles with the phenomenology.

There are two possible scenarios for the relation between the chiral symmetry restoration and the deconfinement phase: If confinement is not necessary for chiral symmetry breaking, i.e. there is no need for a confining propagator in the gap equation, the dynamical gluon mass at zero momentum has to be constrained to the interval  $177 \lesssim m_g(\text{MeV}) \lesssim 204$ . Even more, we argue that it has to be closer to the lower limit to obtain a dynamical quark mass according to phenomenology  $m_g \approx 180\text{MeV} \rightarrow M \approx 300\text{MeV}$ . However, this re-

sult contradict the theoretical bound found in reference [26], where the condition  $m_g > 1.2\Lambda_{QCD}$  is necessary to ensure the positivity of the imaginary part of the gauge boson propagator. To clarify this scenario, simulations in the lattice and accurate estimations of  $m_g$  would be necessary. On the other hand, if a confinement propagator is a necessary ingredient into the gap equation, we can find CSB even for higher values of  $m_g$  because confinement is driving most of the quark mass generation. However, the confinement sector is not sufficient, because it is only when we consider the complete gap equation that a good amount of CSB is reached for the phenomenological value of the gluon mass ( $m_g \approx 600\text{MeV} \rightarrow M = 211\text{MeV}$ ).

A final conclusion concerning the appearance of the multivacuum states is that in any case, the non-perturbative effects (running gluon mass or confinement) break the replicas and define a single vacuum, solution of the fermionic gap equation.

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- [1] C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
  - [2] J. Greensite, *Prog. Part. Nucl. Phys.* **51**, 1 (2003).
  - [3] A. M. Polyakov, *Phys. Lett.* **72B**, 477 (1978).
  - [4] K. I. Kondo, *Phys. Rev. D* **82**, 065024 (2010).
  - [5] A. Mocsy, F. Sannino and K. Tuominen, *Phys. Rev. Lett.* **92**, 182302 (2004).
  - [6] A. Bazavov et al., *Phys. Rev. D* **80**, 014504 (2009).
  - [7] M. Cheng et al., *Phys. Rev. D* **74**, 054507 (2006).
  - [8] J. Engels, S. Holtmann and T. Schulze, *Nucl. Phys. B* **724**, 357 (2005).
  - [9] E. Bilgici, C. Gatttringer, E.-M. Ilgenfritz and A. Maas, *JHEP* **0911**, 035 (2009).
  - [10] R. M. Capdevilla, A. Doff and A. A. Natale, *Phys. Lett. B* **728**, 626 (2014).
  - [11] P. de Forcrand and M. D'Elia, *Phys. Rev. Lett.* **82**, 4582 (1999).
  - [12] P. O. Bowman et al., *Phys. Rev. D* **84**, 034501 (2011).
  - [13] P. J. A. Bicudo, J. E. F. T. Ribeiro and A. V. Nefediev, *Phys. Rev. D* **65**, 085026 (2002).
  - [14] A. V. Nefediev and J. E. F. T. Ribeiro, *Phys. Rev. D* **67**, 034028 (2003).
  - [15] P. J. A. Bicudo and A. V. Nefediev, *Phys. Rev. D* **68**, 065021 (2003).
  - [16] K. Wang, S. Qin, Y. Liu, L. Chang, C. D. Roberts and S. M. Schmidt, *Phys. Rev. D* **86**, 114001 (2012).
  - [17] K. Raya, A. Bashir, S. Hernandez-Ortiz, A. Raya and C. D. Roberts, *Phys. Rev. D* **88**, 096003 (2013).
  - [18] J. M. Cornwall, *Phys. Rev. D* **26**, 1453 (1982).
  - [19] A. Cucchieri and T. Mendes, *PoS QCD-TNT* **09**, 031 (2009); *Phys. Rev. Lett.* **100**, 241601 (2008); *Phys. Rev. D* **81**, 016005 (2010).
  - [20] I.L. Bogolubsky, E.-M. Ilgenfritz, M. Mller-Preussker and A. Sternbeck, *Phys. Lett. B* **676**, 69 (2009).
  - [21] D. Binosi and J. Papavassiliou, *Phys. Rept.* **479**, 1 (2009).
  - [22] A. C. Aguilar, D. Binosi and J. Papavassiliou, *Phys. Rev. D* **89**, 085032 (2014); A. C. Aguilar, D. Binosi, D. Ibañez and J. Papavassiliou, *Phys. Rev. D* **89**, 085008 (2014); A. C. Aguilar, D. Binosi and J. Papavassiliou, *JHEP* **1201**, 050 (2012); A. C. Aguilar, D. Binosi and J. 11 Papavassiliou, *Phys. Rev. D* **84**, 085026 (2011); A. C. Aguilar and J. Papavassiliou, *Phys. Rev. D* **81**, 034003 (2010); A. C. Aguilar, D. Binosi, J. Papavassiliou and J. Rodriguez-Quintero, *Phys. Rev. D* **80**, 085018 (2009); A. C. Aguilar, D. Binosi and J. Papavassiliou, *Phys. Rev. D* **78**, 025010 (2008); A. C. Aguilar and J. Papavassiliou, *JHEP* **0612**, 012 (2006).

- [23] J.M. Cornwall, in: Invited Talk at the Conference Approaches to Quantum Chromodynamics, Oberwlz, Austria, September 2008. hep-ph/0812.0359.
- [24] A. A. Natale, P. S. Rodrigues da Silva, Phys. Lett. B **392**, 444 (1997); Phys. Lett. B 390, 378 (1997).
- [25] B. Haeri, M. B. Haeri, Phys. Rev. D **43**, 3732 (1991).
- [26] J. M. Cornwall, Phys. Rev. D **80**, 096001 (2009).
- [27] J. M. Cornwall, Phys. Rev. D **83**, 076001 (2011).
- [28] S. Mandelstam, Phys. Rev. D **20**, 3223 (1979).
- [29] J. M. Cornwall, Mod. Phys. Lett. A **27**, 1230011 (2012).
- [30] J. M. Cornwall, Phys. Rev. D **57**, 7589 (1998).
- [31] A. Doff, F. Machado, A. Natale, Ann. Phys. **327**, 1030 (2012).
- [32] P. Gonzalez, V. Mathieu, and V. Vento, Phys. Rev. D **84**, 114008 (2011).